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ON THE EXISTENCE OF A POINT SUBSET WITH 4 OR 5 OR *k* INTERIOR POINTS

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Abstract

An interior point of a finite planar point set is a point of the set that is not on the boundary of the convex hull of the set. For any integer $k \geq 6$, let $h^*(k)$ be the smallest positive integer such that every planar point set P with no three collinear points and with at least $h^*(k)$ interior points has a subset Q, whose the interior of the convex hull of Q contains exactly 4 or 5 or k points of P. In this paper, we prove that $h^*(6) = 4$ and $h^*(k) = 7$ for all $k \geq 7$.

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1. Introduction

In this paper, we consider on finite planar point sets such that no three points are collinear. In 1935, Erdös and Szekeres [3] posed a problem: For any integer $k \geq 3,$ determine the smallest integer $f(k) > 0$ such that and finite point set of at least $f(k)$ points has a subset of k points, whose the convex hull contains exactly k vertices. In 2001, Avis et al. [2] posed a problem: For any integer $k \geq 1,$ determine the smallest integer $g(k) > 0$ such that any finite point set P of at least $g(k)$ interior points has a subset Q, whose the interior of the convex hull of Q contains exactly k points in P. Moreover, they posed other problem: For any integer $k \geq 3,$ determine the smallest integer $h(k) > 0$ such that any finite point set P of at least $h(k)$ interior points has a subset Q, whose the interior of the convex hull of Q contains exactly k or $k + 1$ points in P. We known that $h(3) = 3$ (see [2]) and $h(4) = 7$ (see [1]). In 2009, Wei and Ding [4] proved that any point set P with 3 vertices and 9 interior points has a subset with 5 or 6 interior points of P.

In this paper, we pose an interior point problem: For any integer $k \geq 6,$ determine the smallest integer $h^*(k) > 0$ such that any finite point set P of at least $h^*(k)$ interior points has a subset Q, whose the interior of the convex hull of Q contains exactly 4 or 5 or k points in P. We show that $h^*(6) = 4$ and $h^*(k) = 7$ for all $k \geq 7$.

2. Preliminaries

An *interior point* of a finite planar point set is a point of the set that is not on the boundary of the convex hull of the set. Let P be a finite planar point set such that no three points are collinear. Let $Q \subseteq P$. We denote notations as follows:

$\nu(P) :=$ the number of all vertices in P;

$\text{int}\, CH(P) :=$ the interior of the convex hull of P;

$I(P) :=$ the set of interior points of P;

$i(P) :=$ the number of all interior points of P;

and $i^*(Q) :=$ the number of elements in $I(P) \cap \operatorname{int} CH(Q)$, where $Q \subseteq P$.

The set P is called a *deficient point set of type* $P(m, s, k)$, denoted by $P = P(m, s, k)$, if $\nu(P) = m$, $i(P) = s$, and $i^*(Q) \notin \{4, 5, k\}$ for all $Q \subseteq P$.

For any integer $k \geq 6$, we let $h^*(k)$ be the smallest integer such that every planar point set P with no three collinear points and with at least $h^*(k)$ interior points has a subset Q, whose $\operatorname{int} CH(P)$ contains exactly 4 or 5 or k points of P. For each $k \geq 6$,

$$h^*(k) = \min\{s : i(P) \geq s \Rightarrow \exists Q \subseteq P \text{ s.t. } i^*(Q) \in \{4, 5, k\}\}.$$

Proposition 2.1 [1]**.** 7 *is the smallest integer such that any finite point set P of at least* 7 *interior points has a subset* Q, *whose the interior of the convex hull of* Q *contains exactly* 4 *or* 5 *points in P.*

Lemma 2.2. *Let* k *be a positive integer such that* $k \geq 6$. *Then* $h^*(k) \leq 7$.

Proof. Let P be a finite planar point set such that $i(P) \geq 8$. By Proposition 2.1, there is a subset Q of P such that $i^*(Q) \in \{4, 5\}$. Then $i^*(Q) \in \{4, 5, k\}$. Hence, $h^*(k) \leq 7$.

3. Results

In this section, we prove the results: (i) 4 is the smallest integer such that any finite point set P of at least 4 interior points has a subset Q, whose the interior of the convex hull of Q contains exactly 4 or 5 or 6 points in P, and (ii) 7 is the smallest integer such that any finite point set P of at least 7 interior points has a subset Q, whose the interior of the convex hull of Q contains exactly 4 or 5 or k points in P, where $k \geq 7$.

Theorem 3.1. *Let k be a positive integer such that $k \geq 7$. Then* $h^*(k) = 7$.

Proof. By Lemma 2.2, it follows that $h^*(k) \leq 7$. To prove that $h^*(n) \geq 7$, it suffices to show the existence of a deficient point set of type $P(3, 6, k)$. Next, we construct a deficient point set P of type $P(3, 6, k)$ as shown in Figure 1. For all $Q \subseteq P$, we can see that $i^*(Q) \notin \{4, 5, k\}$. Therefore, $h^*(k) = 7$.

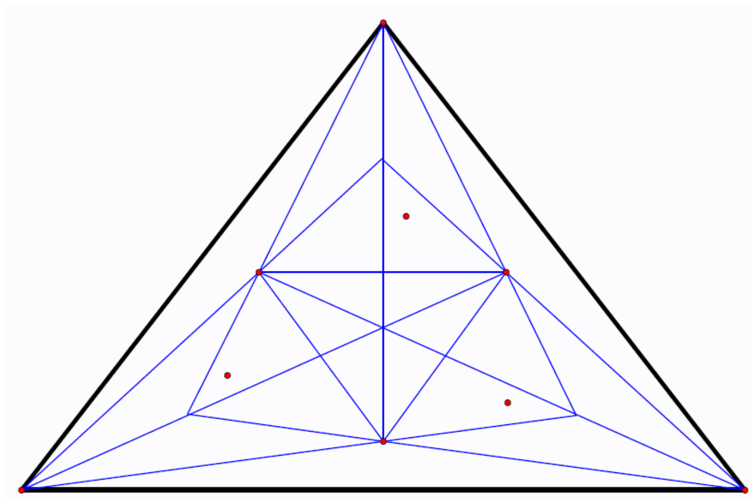

Figure 1. A deficient point set $P = P(3, 6, k)$, where $k \geq 7$.

Theorem 3.2. $h^*(6) = 4$.

Proof. By Lemma 2.2, it follows that $h^*(6) \leq 7$.

If P is a finite planar point set such that $i(P) = 3$, then there is not a subset Q of P such that $i^*(Q) \in \{4, 5, 6\}$. Thus, $4 \leq h^*(6) \leq 7$.

Assume that P is a finite planar point set such that $i(P) \in \{4, 5, 6\}$. Choose $Q = P$. Then $i^*(Q) \in \{4, 5, 6\}$. Thus, $h^*(6) \leq 4$. Therefore, $h^*(6) = 4$.

Corollary 3.3. 3 *is the smallest integer such that any finite point set* P *of at least* 3 *interior points has a subset* Q, *whose the interior of the convex hull of* Q *contains exactly* 3 *or* 4 *or* 5 *or* 6 *points in* P.

Proof. Let P be a finite planar point set and

$$h = \min\{s : i(P) \geq s \Rightarrow \exists Q \subseteq P \text{ s.t. } i^*(Q) \in \{3, 4, 5, 6\}\}.$$

By Theorem 3.2, $h^*(6) = 4$. Then $h \leq 4$. If $i(P) = 3$, then we choose $Q = P$ and then $i^*(Q) \in \{3, 4, 5, 6\}$. Thus, $h \leq 3$. If $i(P) = 2$, then there is not a subset Q of P such that $i^*(Q) \in \{3, 4, 5, 6\}$. Thus, $h > 2$. Hence, $h = 3$.

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